Recovery of the time-evolution equation of time-delay systems from time series

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We present a method for time series analysis of both scalar and nonscalar time-delay systems. If the dynamics of the system investigated is governed by a time-delay-induced instability, the method allows one to determine the delay time. In a second step, the time-delay differential equation can be recovered from the time series. The method is a generalization of our recently proposed method suitable for time series analysis of *scalar* time-delay systems. The dynamics is not required to be settled on its attractor, and this also makes transient motion accessible to the analysis. If the motion actually takes place on a chaotic attractor, the applicability of the method does not depend on the dimensionality of the chaotic attractor—one main advantage over all time series analysis methods known until now. For a demonstration, we analyze time series, which are obtained with the help of the numerical integration of a two-dimensional time-delay differential equation. After having determined the delay time, we recover the nonscalar time-delay differential equation from the time series, in agreement with the ''original'' time-delay equation. Finally, possible applications of our analysis method in such different fields as medicine, hydrodynamics, laser physics, and chemistry are discussed. [S1063-651X(97)14110-1]

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I. INTRODUCTION

Time-delay differential equations have been widely proposed to account for the observed oscillatory, chaotic, or hyperchaotic motion of dynamical systems. Most of the attention has been devoted to scalar time-delayed models [1–9]. Since the pioneering work of Farmer [2], it has been well established that scalar time-delay differential equations are able to exhibit high-dimensional chaotic attractors with many positive Lyapunov exponents and, therefore, are prominent examples to illustrate the chaotic hierarchy [10]. Since then, scalar time-delay equations, especially the well-studied Mackey-Glass system [1], have been used as model systems to produce high-dimensional chaotic time series [11]. In the case of small delay times, the resulting low-dimensional chaotic dynamics is accessible to time series analysis with the help of well-established methods [12–14]. For example, the fractal dimension and the Lyapunov exponents of the chaotic attractors can be estimated. In the case of large delay times, where the dynamics is high-dimensional chaotic, these methods run into severe problems [15].

A first step toward the time series analysis of time-delay systems was performed by Fowler and Kember [16], who showed how ''smart embeddings'' can indicate the presence of an underlying scalar time-delay system for any delay time. Later, we introduced a time series analysis method in order to verify the existence of an underlying time-delay system [17–20]. If the dynamics is governed by a scalar time-delay differential equation, we show that the delay time and the time-delay differential equation can uniquely be recovered from the time series. The method does not put any restriction on the dimensionality of the dynamics analyzed, opening up a door toward a time series analysis of high-dimensional chaotic motion in time-delay systems. Furthermore, the method

does not require the motion to be settled on its attractor. The method turned out to be practically insensitive against additional noise, hence providing a well-suited tool for the analysis of experimental time series. So far, we have successfully applied the method to time series taken from an electronic oscillator [17] and a computer experiment [19]. Although some of the time-delay models investigated are indeed scalar ones, time-delay systems are, in general, nonscalar. Nonscalar time-delay differential equations have been proposed in such different fields as arrays of coupled oscillators [21,22], laser physics [23–25] physiology [26–28], hydrodynamics [29], and chemistry [30] to account for the observed unstable and chaotic dynamical behavior. Additionally, several models with multiple delay times have been investigated [31–33] In this paper, we present a generalization of the time series analysis method proposed in Refs. [17–19]. Herewith, we are able to identify nonscalar time-delay systems and, therefore, to verify the existence of an underlying time-delay-induced instability. The method allows the recovery of the nonscalar time-delay differential equation from the time series. This paper is organized as follows. In Sec. II, the basic idea of the time series analysis method is presented. Also, adequate measures to determine the delay time from the time series are discussed. In Sec. III we illustrate the method by applying it to a two-dimensional time-delay system, the trajectories of which are computed numerically. We show that, while the delay time can still be determined even with a scalar ansatz, the existence of an underlying time-delay system cannot be verified. The latter is accomplished with the help of a nonscalar ansatz, which additionally allows us to recover the time-delay equation from the time series. Finally, possible applications to experimental systems are discussed in Sec. IV.

II. THE BASIC IDEA OF THE ANALYSIS

We consider an N-dimensional time-delay differential equation

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$$\dot{\vec{y}}_0(t) = \hbar(\vec{y}_0(t), \vec{y}_{\tau_0}(t)),$$

$$\vec{y}_{\tau_0}(t) = \vec{y}_0(t - \tau_0),$$

(1)

with the initial condition

$$\vec{y}_0(t) = \vec{y}_i(t), \quad -\tau_0 \leq t \leq 0.$$

(2)

The time evolution of $\vec{y}_0$ at time t does not depend only on its present state $\vec{y}_0(t)$, but also on a state in the past, $\vec{y}_{\tau_0}(t)$, which introduces nonlocal correlations in time. The state of the system is uniquely defined by N functions on an interval of length τ_0. Therefore, the phase space of system (1) is $\mathcal{C}_{\tau_0}^N$, where $\mathcal{C}_{\tau_0}$ is the space of continuous functions on the interval $[-\tau_0, 0]$ and the phase space has to be considered as being infinite dimensional.

The trajectory in the infinite-dimensional phase space $\mathcal{Y}(\tau, t) \in \mathcal{C}_{\tau_0}^N$ can be recovered from its projection $\vec{y}_0(t)$ without loss of information,

$$\mathcal{Y}(\tau, t) = \vec{y}_0(t - \tau), \quad -\tau_0 \leq \tau \leq 0.$$

(3)

We emphasize that construction (3) of the trajectory in phase space is exact, and can be accomplished for all values of the control parameters of the time-delay equation. The N time series $\vec{y}_0(t)$ encompasses the complete information about the dynamics of the system in the infinite-dimensional phase space $\mathcal{C}_{\tau_0}^N$, no matter which dynamical state is realized by the time-delay system. Equation (3) holds for transient motions as well as for motions on chaotic attractors of arbitrary dimension. In general, it is expected that the dynamics of an infinite-dimensional system cannot be reconstructed from a finite number of time series. For instance, to construct the trajectory in the phase space of spatial systems, the dynamics of which is governed by nonlinear partial differential equations, an increasing number of time series with increasing complexity of the dynamics is required.

Additionally, the time derivative $\dot{\mathcal{Y}}(\tau, t) \in \mathcal{C}_{\tau_0}^N$ of the trajectory in phase space can be estimated, in principle, as

$$\dot{\mathcal{Y}}(\tau, t) = \dot{\vec{y}}_0(t - \tau), \quad -\tau_0 \leq \tau \leq 0,$$

(4)

though, in practice, the estimation of time derivatives can be the source of severe errors.

In the case of ordinary as well as time-delay differential equations, the time derivative of the trajectory in phase space is functionally related to the trajectory in phase space via the time-evolution equation. It is the specific property of time-delay systems, though, that only a restricted number of coordinates are correlated via the time-evolution equation (1), namely, the $2N$ coordinates $(\vec{y}_0(t), \vec{y}_{\tau_0}(t))$ taken from the trajectory $\mathcal{Y}(\tau, t)$ in phase space, with the N coordinates $\dot{\vec{y}}_0(t)$ taken from the time derivative $\dot{\mathcal{Y}}(\tau, t)$ of the trajectory.

The basic idea of the time series analysis method presented is to test whether the time-delay equation (1) can be constructed, the solution of which is given by the observed time series. In order to accomplish this task, in general, one has to construct a phase space with the help of the observed time series. In infinite-dimensional systems, one would ex-

pect that an increasing number of time series with increasing complexity of the dynamics is required. We argued in Sec. I that this is not true for time-delay systems, but the observation of N time series is sufficient to construct the trajectory in the infinite-dimensional phase space of an N-dimensional time-delay system, no matter which dynamical state is realized. Second, it is the specific property of time-delay systems that only a restricted number of coordinates are correlated via the time-evolution equation (1), namely, the $2N$ coordinates $(\vec{y}_0(t), \vec{y}_{\tau_0}(t))$ taken from the phase space and one of its time derivatives $\dot{\vec{y}}_0(t)$. Therefore, it is not necessary to analyze the dynamics in the infinite-dimensional phase space, to verify the existence of an underlying time-delay system. It is sufficient to show the existence of a functional relationship (1). To this end, we analyze the dynamics in a $3N$-dimensional space, which is spanned by the coordinates $(\vec{y}_0, \vec{y}_{\tau_0}, \dot{\vec{y}}_0)$. The dynamics of a time-delay system in the $3N$-dimensional space is restricted to a $2N$-dimensional hypersurface, which is given by the time-evolution equation (1). The hypersurface (1) is defined that to any value $(\vec{y}_0(t), \vec{y}_{\tau_0}(t))$ there is a unique value of the time derivative $\dot{\vec{y}}_0(t)$ for all times t.

The idea of the time series analysis method presented in this paper is to test the existence of such a hypersurface for a given time series. Starting with N scalar time series, which have been taken from the system to be investigated, we hypothesize, at first, that the dynamics is governed by an N-dimensional time-delay system

$$\dot{\vec{y}}_0(t) = \vec{h}_r(\vec{y}_0(t), \vec{y}_\tau(t)),$$

$$\vec{y}_\tau(t) = \vec{y}_0(t - \tau),$$

(5)

with an unknown function $\vec{h}_r$ and an unknown delay time τ, both of which will be determined in the subsequent analysis, if ansatz (5) turns out to be successful. Then we take the values of $(\vec{y}_0, \vec{y}_\tau)$ and $\dot{\vec{y}}_0$ from the time series, and analyze its dynamics in the $3N$-dimensional space, which is spanned by the coordinates $(\vec{y}_0, \vec{y}_\tau, \dot{\vec{y}}_0)$. If the coordinates of the trajectory $(\vec{y}_0(t), \vec{y}_\tau(t), \dot{\vec{y}}_0(t))$ are functionally correlated via Eq. (5), the hypothesis that the system is governed by a time-delay equation with the delay time τ has been verified. If the projected trajectory $(\vec{y}_0, \vec{y}_\tau, \dot{\vec{y}}_0)$ does not fulfill condition (5), the hypothesis has to be rejected. This is a unique criterion to determine the delay time from the time series. Additionally, the time-delay differential equation can be constructed by analyzing the functional relationship (5), which exactly gives the function $\hbar_r$. Obviously, the only requirement remains that the system dynamics fulfills the time-evolution equation (5), which even is true for all kinds of transient motion as well as for the motion on chaotic or hyperchaotic attractors of arbitrary dimension. Therefore, the method permits us to analyze high-dimensional chaotic dynamics of time-retarded systems, which is not accessible to the fractal dimension analysis. If the system possesses several coexisting attractors [7], the method is applicable to the dynamics on every attractor.

We have argued above that the existence of an underlying time-delay system can be verified by proving the existence of a $2N$-dimensional hypersurface $\bar{h}_r$ in form (5) in the $3N$-dimensional space, which is spanned by the coordinates $(\vec{y}_0, \vec{y}_\tau, \dot{\vec{y}}_0)$. Therefore, it is crucial to apply adequate measures, which enable us to identify such a hypersurface by analyzing the time series. All measures proposed so far [16–19] rely solely on the fact that, if the trajectory $(\vec{y}_0(t), \vec{y}_\tau(t), \dot{\vec{y}}_0(t))$ is correlated via Eq. (5), the dimensionality of the trajectory is reduced. To our knowledge, this was realized for the first time by Fowler and Kember [16], who analyzed the dynamics of the Mackey-Glass equation. They applied an embedding of the time series in a three-dimensional space with two time-delayed coordinates. The delay time of the first coordinate has been chosen to be small. The delay time of the second coordinate was taken as variable. The authors of Ref. [16] stated that if the delay time of the second coordinate equals the delay time of the time-delay system, the trajectory lies ''close to a surface''. Fowler and Kember applied a singular value fraction to detect the decrease in dimensionality. As was correctly mentioned by them, the singular value fraction is not a good tool if the surface is folded. The latter must be considered as the general case.

Recently, [17–20] we proposed a time series analysis method for scalar time-delay systems only. There, we showed that the trajectory in $(y_0, y_{\tau_0}, \dot{y}_0)$ space is restricted to a two-dimensional surface. The reduction in dimension was detected by intersecting the projected trajectory with a surface $k(y_0, y_\tau, \dot{y}_0) = 0$. The intersection points must be on a curve, if the projected trajectory is correlated via Eq. (5). We detected such a behavior by ordering the intersection points with respect to one coordinate and drawing a polygon line, which connects all intersection points. The length of the polygon line has been taken as a measure for the alignment of the intersection points. This simple method correctly determined the delay time of scalar time-delay systems, but there is no straightforward generalization for nonscalar time-delay systems.

For this reason, we apply another method, the basic idea of which is the following: If the trajectory of an N-dimensional time-delay system in the $(\vec{y}_0, \vec{y}_{\tau_0}, \dot{\vec{y}}_0)$ space, where τ_0 is assumed to be the correct value of the delay time, is correlated via Eq. (5), the trajectory is restricted to a hypersurface. Therefore, most parts of the $(\vec{y}_0, \vec{y}_{\tau_0}, \dot{\vec{y}}_0)$ space are not visited by the trajectory. If the trajectory is viewed in any other space, for instance, if the value of the delay time is not chosen properly, the projected trajectory is expected to visit ''more'' parts of the space. Therefore, we compute the filling factor of the projected trajectory by covering the $(\vec{y}_0, \vec{y}_\tau, \dot{\vec{y}}_0)$ space with P^{3N} equally sized hypercubes. The filling factor is the number of hypercubes, which are visited by the projected trajectory, normalized to the total number of hypercubes, P^{3N}. The filling factor is computed under variation of τ. The existence of an underlying time-delay-induced instability induces a local minimum in the filling factor.

Fowler and Kember [16] already suggested a fractal dimension analysis of the projected trajectories. The fractal

dimension analysis has some severe drawbacks: First, a fractal analysis requires a large number of data points, because for the determination of a fractal dimension it is crucial to resolve the geometrical object under investigation on different ''length scales.'' Second, the fractal analysis is computationally intensive and, in practice, sensitive to additional noise. All those measures rely on the fact that the trajectory is restricted to a hypersurface, if the projected trajectory is correlated via Eq. (5). It has been argued above that the dimension reduction is not a sufficient criterion for the verification of an underlying time-delay system. The existence of the functional relationship (5) has to be shown separately.

III. TIME SERIES ANALYSIS OF A TWO-DIMENSIONAL TIME-DELAY SYSTEM—A NUMERICAL EXAMPLE

The applicability of the method for time series analysis is demonstrated with the help of a computer experiment. A nonscalar time-delay system is integrated numerically. Details of the numerical integration are reported in the Appendix. We will show that, although a scalar ansatz also leads to a local minimum in the filling factor, the scalar ansatz must be rejected, because it is not possible to find a surface, which is given by an equation of form (5) with the help of a scalar ansatz. In a second step, we will identify the system as a nonscalar time-delay system by verifying the existence of such a surface with the help of a nonscalar ansatz. Finally, the time-delay differential equation will be recovered from the time series.

We consider the two-dimensional time-delay differential equation, which has been chosen to serve its demonstrational purpose best:

$$\dot{u} = -v + f(u_{\tau_0}),$$
$$\dot{v} = g(u, v), \tag{6}$$

with the initial conditions

$$u(t) = u_i(t), \quad -\tau_0 \leq t \leq 0,$$
$$v(t=0) = v_i.$$

The functions f and g are given by

$$f(u_{\tau_0}) = \frac{a u_{\tau_0}}{1 + u_{\tau_0}^{10}}, \tag{7}$$

$$g(u, v) = -\frac{1}{T}(v - u). \tag{8}$$

Equation (6) has some similarity to the Mackey-Glass system. The dependence of $\dot{u}$ on the time-delayed value u_{τ_0} is the same as it is the case in the Mackey-Glass system. But while the dependence of $\dot{u}$ on u induces exponential relaxations in the Mackey-Glass system, in Eq. (6) it is similar to a damped oscillator. There are two limits of the control parameter space (a, T, τ_0), where the dynamics of Eq. (6) is well known. For $a \to 0$, system (6) reduces to a damped harmonic oscillator. For $T \to 0$, the time scale of v is much faster compared to the time scale of u. The variable v, then,
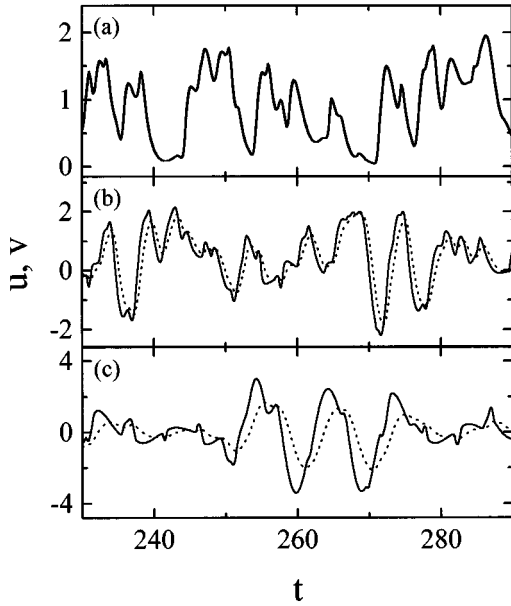
FIG. 1. Time series of system (6) obtained with the help of numerical integration ($a=3.00$, $\tau_0=20.00$) for different values of T: (a) $T=0.10$, (b) $T=0.60$, and (c) $T=1.90$.
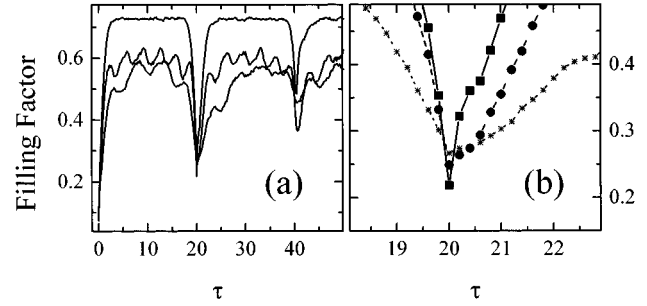

FIG. 2. (a) Filling factor of the time series $u(t)$ of system (6) with $a=3.00$, $\tau_0=20.00$ for different values of T (lower curve: $T=0.10$; middle curve: 0.60; upper curve: 1.90). We used 100 000 data points for the filling factor analysis, which were taken out of a time series of 1 000 000 data points. In (b), a blowup of the filling factor in the vicinity of the delay time $\tau=\tau_0$ is shown (squares: $T=0.10$; circles: $T=0.60$; stars: $T=1.90$).

adiabatically follows variable u, and the dynamics of Eq. (6) resembles that of the Mackey-Glass system. Here a and τ_0 are chosen such that in the scalar limit ($T \to 0$) the dynamics is high-dimensional chaotic ($a=3$, $\tau_0=20$). T is varied from 0.10 to 1.90. Note that system (6) can be transformed to a scalar integrodifferential equation for the variable u by integrating the second equation with the help of the method of varying coefficients.

We present three time series of u and v for different T in Fig. 1. In Fig. 1(a), it is clearly seen how the variable v, for $T=0.10$, follows the variable u and the dynamics of u resembles that of a Mackey-Glass system. The values of (u,v) are positive for all times. We mention at this point that system (6) is invariant under the transformation $(u,v) \to (-u,-v)$. Therefore, there exists another attractor with negative values of (u,v). In Figs. 1(b) and 1(c), we observe that the two coexisting attractors are merged. Variable v no longer follows variable u, but it develops an independent dynamics. In these cases, the system reveals its non-scalar nature.

Now, we analyze these time series with the help of a filling factor analysis. At first, we choose the scalar ansatz

$$\dot{u}=h_r(u,u_\tau) \qquad (9)$$

for the analysis of the time series $u(t)$ with an unknown delay time τ and an unknown function h_r. We analyze the time series in a three-dimensional space, which is spanned by the coordinates $(u,u_\tau,\dot{u})$ with a variable value of τ. Then, the filling factor of the time series is determined under variation of τ.

The results of the filling factor analysis are presented in Fig. 2 for different values of T. The filling factor is minimal for small values of τ as a result of local correlations in time. The filling factor increases for increasing τ, which is a fingerprint of the chaotic nature of the motion, and eventually

reaches a maximal value. For $\tau=\tau_0=20.00$, a local minimum of the filling factor is observed for all values of T. This decrease in the filling factor is due to the nonlocal correlations in time induced by the time delay. An additional local minimum appears at $\tau=2\tau_0$. For high enough values of T, though, other regularly spaced local minima in the filling factor appear, the period of which is equal to the oscillations of the underlying damped oscillator. In Fig. 2(b), a blowup of the τ-dependent filling factor in the vicinity of the delay time $\tau_0=20.00$ is shown. Clearly, the local minimum appears for all values of T considered here, but it is less pronounced for increasing T, because the character of the time-delay system then becomes more and more nonscalar. As emphasized above, the reduction in dimension is a necessary, but not a sufficient condition for the verification of a scalar time-delay system. The existence of a surface in form (9) has to be checked as well.

To this end, we apply an intersection of the time series $u(t)$ with the plane $\dot{u}(t^i)=0$. The values of the coordinates $u^i=u(t^i)$ and $u^i_{\tau_0}=u_{\tau_0}(t^i)$ are recorded. They have to be correlated according to the scalar ansatz (9),

$$h_r(u^i,u^i_{\tau_0})=0. \qquad (10)$$

Therefore, the points $(u^i,u^i_{\tau_0})$ have to lie on a curve, if the dynamics is governed by a scalar time-delay equation. The intersection points $(u^i,u^i_{\tau_0})$ for $\tau=\tau_0=20.00$ and different values of T are shown in Fig. 3. For small values of T, the time series $v(t)$ follows the time series $u(t)$. In this case, the dynamics of variable u is close to the dynamics of the Mackey-Glass system. Such a behavior can be seen in Fig. 3(a), where the intersection yields a geometrical object, which is close to being a one-dimensional curve. The inset of Fig. 3(a) shows a blowup of the intersection points. Clearly, the alignment of the intersection points is not perfect, as a result of the nonscalar nature of system (6). Nevertheless, the scalar ansatz (9) would be a good approximation for small values of T. For higher values of T, system (6) reveals its nonscalar nature. Therefore, in Figs. 3(b) and 3(c), the distribution of the intersection points becomes cloudy. That means, a smooth functional relationship (9) cannot be found
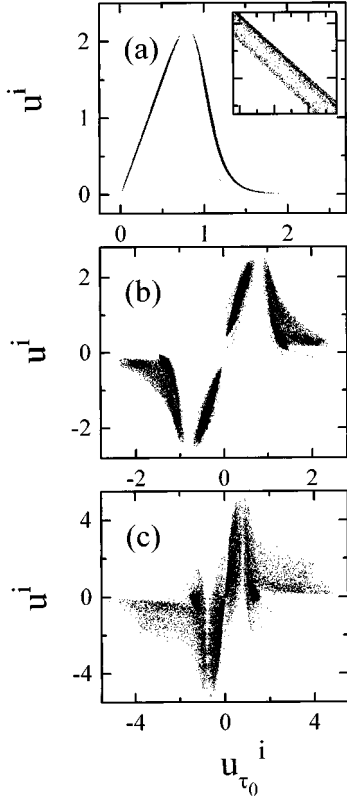
FIG. 3. Intersection points of the time series $u(t)$ for $a = 3.00$, and $\tau_0 = 20.00$ and different values of T: (a) $T = 0.10$, (b) $T = 0.60$, and (c) $T = 1.90$. The analysis has been conducted with a time series of 1 000 000 data points.
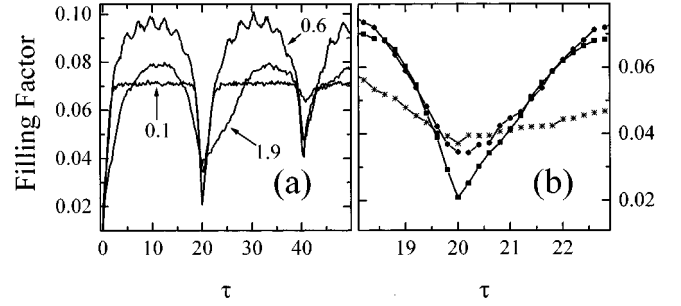

FIG. 4. (a) Filling factor of the two time series $(u(t), v(t))$ of system (6) with $a = 3.00$, and $\tau_0 = 20.00$ for different values of T ($T = 0.10$, 0.60, and 1.90; the values of T are indicated in the figure). We used 100 000 data points for the filling factor analysis, which were taken out of a time series of 1 000 000 data points. In (b) a blowup of the filling factor in the vicinity of the delay time $\tau = \tau_0$ is shown (squares: $T = 0.10$; circles: $T = 0.60$; stars: $T = 1.90$).

The delay time τ and the functions f_r and g_r are still unknown, and will be determined in the following. First, we perform a filling factor analysis in the same spirit as has been done in the scalar case. But now the two time series $(u(t), v(t))$ have to be projected to a six-dimensional space which is spanned by the coordinates $(u, u_\tau, \dot{u}, v, v_\tau, \dot{v})$. The six-dimensional space is covered with equally sized hypercubes and the number of hypercubes which have been visited by the trajectory is counted under variation of τ.

The results are presented in Fig. 4 for different values of T, namely, $T = 0.10$, 0.60, and 1.90. The minimum in the filling factor for $\tau = \tau_0 = 20.00$ is well detected for all values of T. In Fig. 4(b), we show a blowup of the τ-dependent filling factor in the vicinity of the delay time τ_0 of system (6). Clearly, the local minimum for $\tau = \tau_0$ is detected for all values of T.

As argued above, the existence of the functional relationship (5) has to be shown, in order to verify the underlying time-delay-induced instability. The special form of the ansatz [Eqs. (13) and (14)] together with the time-evolution equations (6) allows for a convenient way of proving the existence of function (5). We emphasize, though, that, in general, it is expected to be more troublesome. We apply an intersection with the help of the condition $v(t^i) = 0$. If the nonscalar ansatz, Eqs. (13) and (14), is successful, the values $\dot{u}^i = \dot{u}(t^i)$ and $u^i_{\tau_0} = u_{\tau_0}(t^i)$ have to be correlated via

$$\dot{u}^i = f_r(u^i_{\tau_0}). \tag{15}$$

in these cases and the scalar ansatz has to be rejected. Nevertheless, in the considered example of a two-dimensional time-delay system the filling factor analysis was successful in the framework of a scalar ansatz, in the sense that the τ-dependent filling factor showed local minima for the correct values of the delay time.

Now, we will analyze the time series for higher values of T with the help of a nonscalar ansatz. The general two-dimensional ansatz is

$$\dot{u} = h_{r,1}(u, u_\tau, v, v_\tau), \tag{11}$$

$$\dot{v} = h_{r,2}(u, u_\tau, v, v_\tau), \tag{12}$$

with an unknown delay time τ and two unknown functions $h_{r,1}$, and $h_{r,2}$. The analysis has to be conducted in a six-dimensional space, in which the dynamics is restricted to a four-dimensional hypersurface. The hypersurface is given by the functions $h_{r,1}$, and $h_{r,2}$, which can be determined with the help of adequate fitting procedures, for instance, a least-squares-fit in the framework of a presupposed model. In this paper, we chose a more restrictive ansatz for demonstrational purposes,

$$\dot{u} = -v + f_r(u_\tau), \tag{13}$$

$$\dot{v} = -g_r(u, v). \tag{14}$$

Plotting $\dot{u}^i$ versus $u^i_{\tau_0}$, as is shown in Fig. 5(a), the existence of the smooth function f_r is verified. We compare the reconstructed function f_r (open circles) with the function f (line) of the time-delay differential equation (6). The coincidence is good and the $(\dot{u}^i, u^i_{\tau_0})$ plot can be used to recover the function f from the time series. We emphasize that no parameter has been adjusted to compare the function f with its recovery f_r in Fig. 5(a).

In the next step, the functional relationship between $\dot{v}^i$ and u^i is investigated. We use the same intersection condition $v(t^i) = 0$ as above. According to the nonscalar ansatz, Eqs. (13) and (14), the coordinates of the intersection points are correlated via
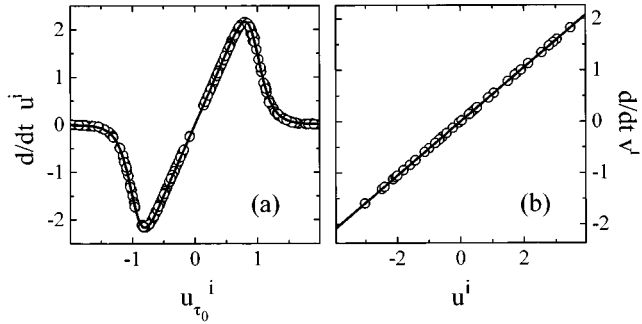
FIG. 5. Recovery of the time-delay differential equation from the time series: (a) Comparison of the data points $(\dot{u}^i, u^i_{\tau_0})$, which are shown as open circles, with the function f (line). For clarity, only 300 data points $(\dot{u}^i, u^i_{\tau_0})$ are shown. (b) Comparison of the data points $(\dot{v}^i, u^i)$, which are shown as open circles, with the function $g(u^i, 0)$ (line). For clarity, only 40 data points $(\dot{v}^i, u^i)$ are shown.

$$\dot{v}^i = -g_r(u^i, 0). \tag{16}$$

Plotting $\dot{v}^i$ versus u^i yields the function $g_r(u^i, 0)$. We compare the functions $g(u^i, 0)$ and $g_r(u^i, 0)$ in Fig. 5(b). Again, the correspondence is good and the $(\dot{v}^i, u^i)$ plot can be used to recover the function $g_r(u^i, 0)$ from the time series. The recovery of the function $g_r(0, v^i)$ is to be done in the same spirit, and is not shown here. Obviously, the existence of the functions f_r and g_r has been proven and a two-dimensional time-delay system has been identified by analyzing the time series.

IV. APPLICATIONS

Finally, we would like to discuss possible applications of the present method for time series analysis. We emphasize that our method does not have the restrictions which are inevitable in the embedding techniques necessary for the determination of fractal dimensions of chaotic attractors in phase space. The analysis is not restricted to a low-dimensional chaotic motion. Transients can be analyzed as well. The method is not sensitive to additional noise. Furthermore, it has been shown that the analysis can be performed with a comparably small number of data points [20]. Apparently, we have a well-suited tool for the analysis of experimental time series. If the dynamics of the system to be investigated is governed by a time-delay-induced instability, the method allows for the determination of the delay time and a recovery of the time-delay differential equation. Therefore, it is possible to compare time series of experimental systems with proposed model equations in detail. System parameters can be extracted from the time series analysis, which might be not accessible otherwise.

We speculate that the analysis can be particularly useful in such fields as medicine and biology, where noninvasive techniques are of great importance for obvious reasons. In several experiments on human subjects, which were exposed to time delays of some sort, a qualitative change of the observed dynamics has been verified [1,6,9,26,34], which could be correlated to well-established pathologies. If the observed dynamics is, indeed, induced by a time delay, as proposed, we expect our analysis method to be successful. On the other

hand, we propose to check the validity of the Mackey-Glass system by analyzing suitable time series. The analysis might improve the understanding of the experiments, and possibly allows us to determine important system parameters and serve as a new diagnostic tool.

In a recent paper, Villermaux [29] dealt with the low-frequency oscillations of the velocity field in the "hard turbulence" regime in a closed convection box. The author proposed a two-dimensional time-delay system to describe the dynamics of the disturbances. Our method has the potential to test the validity of the model by analyzing the experimental time series, which possibly can lead to a better understanding of boundary instabilities.

Another important class of a prototype model for chaotic dynamics are laser systems. No wonder that time-delay models have also been investigated in laser physics ([3,23,24,31,33] and references therein). It is the advantage of laser systems with a time delay that they allow precise measurements of time series [25,32]. We find them particularly suitable for our analysis, because we expect the time delay of laser systems to be practically discrete, compared to other experimental systems. The first steps toward the identification of high-dimensional chaotic dynamics of a time-delayed laser system have been taken [20].

V. CONCLUDING REMARKS

In conclusion, we have presented a generalization of a recently proposed method for recovering the time-evolution equation of scalar time-delay systems by analyzing the time series. The method is generalized in the way that it can be applied to nonscalar time-delay systems.

We have shown that an N-dimensional time-delay system can be identified with the help of N time series. The analysis need not be conducted in the infinite-dimensional phase space, instead it is sufficient to analyze the dynamics in a $3N$-dimensional space, in which the dynamics has to be restricted to a $2N$-dimensional hypersurface. This finding gives us a unique criterion to determine the delay time of a nonscalar time-delay system by analyzing the time series. Additionally, the time-delay differential equation of dimension N can be recovered.

We emphasize that we only require the motion to obey the time-delay differential equation. The motion is not required to be located in certain parts of the phase space. If the dynamical system possesses coexisting attractors, the method can be applied to motions on every coexisting attractor. Moreover, the method is also applicable to transient motions. If the motion is on a chaotic or hyperchaotic attractor, the applicability of the analysis method depends neither on the dimensionality nor on the number of positive Lyapunov exponents of the chaotic or hyperchaotic attractor. Therefore, we find that the present method might open up a door toward a time series analysis of high-dimensional chaotic motion in time-delay systems.

We have shown the applicability of the method by analyzing time series which have been obtained with the help of numerically integrating a two-dimensional time-delay differential equation. The system investigated mimics a scalar time-delay system in a certain parameter range, where a scalar ansatz yields a good approximation. Under variation of a

single control parameter, the dynamics increasingly reveals the nonscalar nature of the time-evolution equation. In this case, a scalar ansatz is not sufficient, but, nevertheless, we have successfully analyzed the time series with the help of a nonscalar ansatz. The delay time has been determined from the time series. Finally, we recovered the two-dimensional time-delay differential equation by analyzing the time series. Possible applications for the analysis of dynamical systems in such different fields as medicine, hydrodynamics, laser physics, and chemistry are discussed.

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APPENDIX: NUMERICAL METHODS

The results presented in this paper were computed with the help of a Runge-Kutta algorithm of the fourth order. The fundamental time step was taken to be 0.01. The length of the memory was 2,000 time steps. In all simulations presented, we chose homogeneous initial values of $u(t) = 0.50$, $v(t) = 0.90$, and $-\tau_0 \leq t \leq 0$.

We checked the validity of the numerical integration of system (6) by comparing the results of Runge-Kutta algorithms of different orders, under variation of the time step of the integration, in order to check the validity of the results. Additionally, the computed time series were compared to analytical solutions, which are available in certain parameter ranges.

For the filling factor analysis, we used time series with 1 000 000 data points. The time derivatives were estimated by applying a local parabolic approximation. To compute the τ-dependent filling factor, every tenth point of the time series was taken. Therefore, we conducted the filling factor analysis by analyzing 100 000 data points in three- or six-dimensional space, respectively.

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